1.3 PHYSICAL SYSTEM RESPONSE PROPERTIES

Most of the applications of correlation and spectral density functions considered in this book involve some physical system. A brief review is presented here of the response properties of physical systems that are pertinent to material in later chapters. The emphasis is on mechanical systems, since they are the basis for most of the later illustrations. However, using classical analogies [1.3], the relationships presented here apply equally well to many other physical systems.

1.3.1. Unit-Impulse Response Functions

An ideal physical system is one which (*a*) is physically realizable, (*b*) has constant parameters, (*c*) is stable, and (*d*) is linear, all to be defined shortly. For such an ideal system, the basic response properties of primary interest are given by the response of the system to a delta-function input, called the *unit-impulse response function* or the *weighting function* $h(\tau)$. Specifically, consider a single-input/single-output system with a well-defined input $x(t)$ producing a well-defined output $y(t)$, as shown in Figure 1.7. The unit-impulse response function is given by

$$h(t) = y(t) \qquad \text{when } x(t) = \delta(t) \tag{1.39}$$

where t is time measured from the instant the delta function input is applied. The importance of the unit-impulse response function as a description of the system is due to the following fact: For any arbitrary input $x(t)$, the linear system output $y(t)$ is given by the *superposition* or *convolution* integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \tag{1.40}$$

That is, the response $y(t)$ is given by a weighted linear sum over the entire time history of the input $x(t)$.

A *physically realizable* system cannot respond to an input until that input has been applied. This requires that

$$h(\tau) = 0 \qquad \text{for } \tau < 0 \tag{1.41}$$

Hence for physically realizable systems, the lower limit of integration in Equation 1.40 is zero rather than minus infinity.

Figure 1.7 Ideal single-input / single-output system.

A physical system is said to have *constant parameters* if the unit-impulse response function is not dependent on the time an input is applied, that is,

$$h(t,\tau) = h(\tau) \qquad \text{for } -\infty < t < \infty \tag{1.42}$$

If a physical system has constant parameters, stationary inputs will always produce stationary responses (after switch-on transients decay).

A physical system is said to be *stable* if every possible bounded input produces a bounded response. This condition is assured if

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty \tag{1.43}$$

System stability is required for all the input/output relationships and applications discussed in this book.

A *linear* system is additive and homogeneous. Given two inputs x_1 and x_2 which individually produce outputs y_1 and y_2 per Equation 1.40, the system is *additive* if the input $x_1 + x_2$ produces an output $y_1 + y_2$, and is *homogeneous* if the input cx_1 produces the output cy_1, where c is an arbitrary constant. This essentially means that $h(\tau)$ is not dependent on the input, that is,

$$y(t) = \int_0^{\infty} h(\tau)\, x(t-\tau)\, d\tau \qquad \text{for all } x(t) \tag{1.44}$$

If a system is linear, random inputs with a Gaussian probability distribution (to be defined in Chapter 2) will produce outputs that also have a Gaussian probability distribution.

Linearity is the most likely property of physical systems to be violated in practice. In particular, with random inputs, there is usually a small probability of an instantaneous input so extreme that the system can no longer respond in a directly proportional manner as demanded by the homogeneity requirement. This is an important and difficult problem in those applications which involve extreme-value statistics, for example, the prediction of catastrophic failures of structures under random loading. It may also be a problem in more general applications, such as determining the response properties of structures with nonlinear stiffness and/or damping parameters. Appropriate analysis procedures for many types of nonlinear systems are developed in Chapter 13, and detailed further in Reference 1.4.

1.3.2 Frequency Response Functions

The dynamic properties of physical systems are usually described in terms of some linear transformation of the unit-impulse response function $h(\tau)$ rather than $h(\tau)$ itself. Any one of several such linear transformations might be employed for special applications. For ideal systems, however, a Fourier transformation producing a direct frequency-domain description of the system properties is most desirable from the viewpoint of the applications of concern in this book. The Fourier transform of the unit-impulse response function where $h(\tau) = 0$ for $\tau < 0$ is given by

$$H(f) = \int_0^\infty h(\tau) e^{-j2\pi f\tau}\, d\tau \tag{1.45}$$

and is called the *frequency response function*. The frequency response function is generally a complex number with real and imaginary parts given by

$$H(f) = H_R(f) - jH_I(f) \tag{1.46}$$

$$H_R(f) = \int_0^\infty h(\tau)\cos 2\pi f\tau\, d\tau \qquad H_I(f) = \int_0^\infty h(\tau)\sin 2\pi f\tau\, d\tau$$

For convenience, it will be described throughout this book in terms of complex polar notation as follows:

$$H(f) = |H(f)| e^{-j\phi(f)} \tag{1.47}$$

$$|H(f)| = \left[H_R^2(f) + H_I^2(f)\right]^{1/2} \qquad \phi(f) = \tan^{-1}\left[\frac{H_I(f)}{H_R(f)}\right]$$

The magnitude $|H(f)|$ is commonly referred to as the system *gain factor*, and the phase $\phi(f)$ is called the system *phase factor*. Note that the phase factor is defined so that lag angles are positive, to be consistent with the sign conventions used throughout this book.

The physical interpretation of the frequency response function is straightforward. For an ideal system as described in Section 1.3.1, a sinusoidal input at frequency f will produce a sinusoidal output at exactly the same frequency f. However, the amplitude of the output will generally be different from the input amplitude, and the output will generally be shifted in phase from the input as follows:

$$x(t) = X\sin 2\pi ft \qquad y(t) = Y\sin(2\pi ft - \theta) \tag{1.48}$$

The ratio of the output to input amplitudes equals the system gain factor, and the phase shift between the output and input is the system phase factor at frequency f, that is,

$$|H(f)| = \frac{|Y(f)|}{|X(f)|} \qquad \phi(f) = \theta(f) \tag{1.49}$$

The term *transfer function* is commonly used by engineers to denote the same quantity as the frequency response function of Equation 1.45. However, to be precise, the transfer function should be defined by the *Laplace* transform equation

$$H_1(p) = \int_0^\infty h(\tau) e^{-p\tau} \, d\tau \qquad p = a + jb \tag{1.50}$$

where the real part of p, namely a, is not restricted to be zero. For $a \neq 0$, such Laplace transforms will be different from the Fourier transforms of Equation 1.45. Along the imaginary axis where $a = 0$, by taking $b = 2\pi f$, one obtains $H_1(j2\pi f) = H(f)$. Thus along the imaginary axis, the transfer function is the same as the frequency response function, which helps explain why these terms are often interchanged.

1.3.3 Single-Degree-of-Freedom Systems

To illustrate a frequency response function of common interest, consider the simple mechanical system consisting of a mass m, a spring with spring constant k, and a velocity-dependent damping mechanism with damping coefficient c, as shown in Figure 1.8. A mechanical system of this type is commonly referred to as a single-degree-of-freedom (SDOF) system. Assume the mass is subjected to a force input $F(t)$ producing a displacement response $y(t)$. From Newton's laws, the linear differential equation of motion describing the response of this system is given by

$$m \frac{d^2 y(t)}{dt^2} + c \frac{dy(t)}{dt} + ky(t) = F(t) \tag{1.51}$$

To find the frequency response function of the system, let the input $F(t) = \delta(t)$, a delta function as defined as Section 1.2.4. Then from Equation 1.39, the response $y(t) = h(t)$, and from Equation 1.45, the Fourier transform of

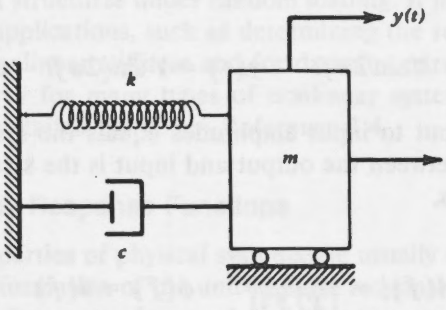

Figure 1.8 Single-degree-of-freedom mechanical system.

the response $Y(f) = H(f)$. Taking Fourier transforms of both sides of Equation 1.51 yields

$$\left[-(2\pi f)^2 m + j2\pi fc + k\right] Y(f) = 1$$

Thus

$$Y(f) = H(f) = \left[k - (2\pi f)^2 m + j2\pi fc\right]^{-1} \tag{1.52}$$

It is convenient to write Equation 1.52 in a different form by introducing two definitions:

$$\zeta = \frac{c}{2\sqrt{km}} \qquad f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1.53}$$

The quantity ζ in Equation 1.53 is called the *damping ratio* of the system and describes the system damping as a fraction of the critical damping c_c. If the mass is displaced from its neutral position and released, c_c is that value of damping just sufficient for the mass to return to its neutral position without further oscillation; for the system in Figure 1.8, $c_c = 2\sqrt{km}$. The quantity f_n in Equation 1.53 is called the *undamped natural frequency* of the system. If the system had no damping and the mass were displaced from its neutral position and released, the system would perpetually oscillate at the frequency f_n. By using the definitions in Equation 1.53, the frequency response function of the system in Equation 1.52 may be written as

$$H(f) = \frac{1/k}{1 - (f/f_n)^2 + j2\zeta f/f_n} \tag{1.54}$$

In terms of the system gain and phase factors defined in Equation 1.47,

$$|H(f)| = \frac{1/k}{\sqrt{\left[1 - (f/f_n)^2\right]^2 + \left[2\zeta f/f_n\right]^2}} \tag{1.55a}$$

$$\phi(f) = \tan^{-1}\left[\frac{2\zeta f/f_n}{1 - (f/f_n)^2}\right] \tag{1.55b}$$

Plots of these gain and phase factors are shown in Figure 1.9.

Two characteristics of the plots in Figure 1.9 are of particular interest. First, the gain factor has a peak at some frequency less than f_n for all cases where $\zeta \leq 1/\sqrt{2}$. The frequency at which this peak gain factor occurs is called the *resonance frequency* of the system. Specifically, it can be shown by minimizing the denominator of $|H(f)|$ in Equation 1.55a that the resonance

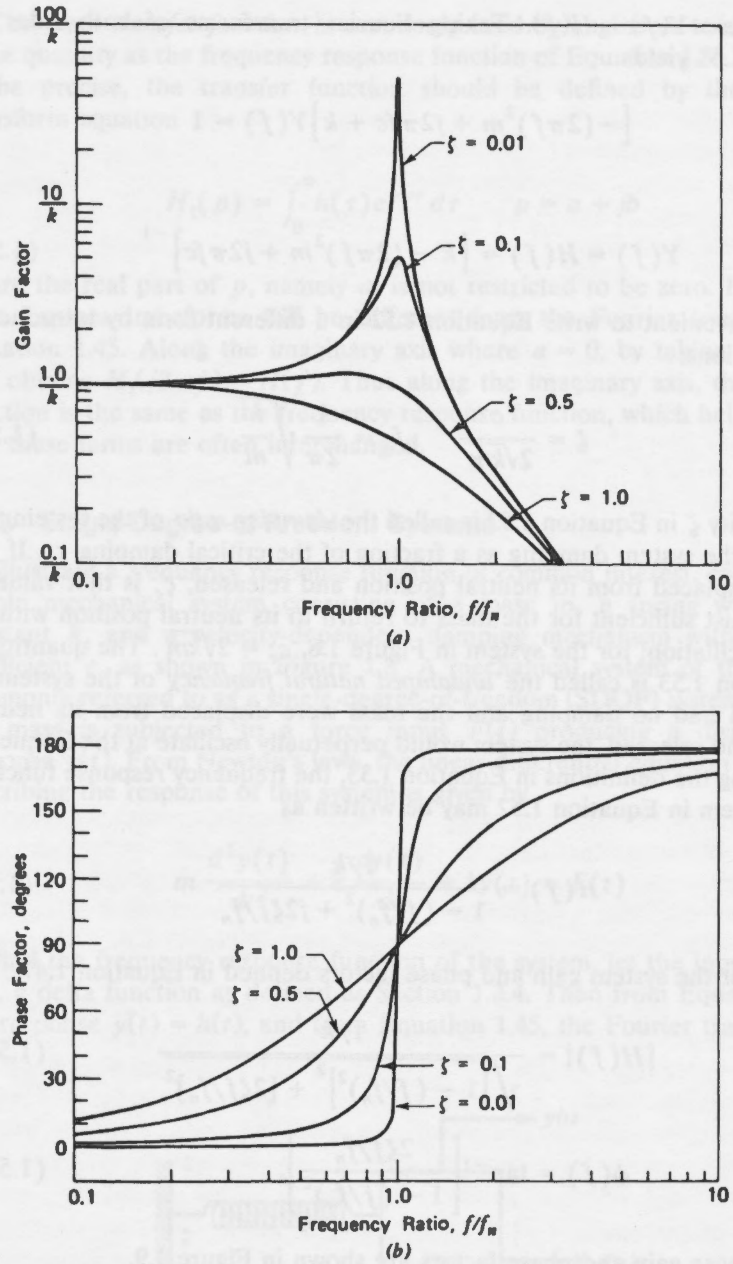

Figure 1.9 Frequency response function of a single-degree-of-freedom system with force input and displacement output. (*a*) Gain factor. (*b*) Phase factor.

frequency, denoted by f_r, is given by

$$f_r = f_n\sqrt{1 - 2\zeta^2} \qquad \zeta^2 \le 0.5 \tag{1.56}$$

and that the peak value of the gain factor which occurs at the resonance frequency is given by

$$|H(f_r)| = \frac{1/k}{2\zeta\sqrt{1 - \zeta^2}} \qquad \zeta^2 \le 0.5 \tag{1.57}$$

Second, the phase factor varies from 0° for frequencies much less than f_n to 180° for frequencies much greater than f_n. The exact manner in which $\phi(f)$ varies between these phase-angle limits depends on the damping ratio ζ. However, for all values of ζ, the phase $\phi(f) = 90°$ for $f = f_n$.

Actual physical systems often have very small values of damping such that $\zeta \ll 1$. For example, mechanical structures generally have damping ratios of $\zeta < 0.05$. Hence it is common in practice to find physical systems with gain factors that display very sharp peaks and phase factors that show rapid 180° phase shifts. Such systems appear, in effect, to be narrow-bandpass filters, and their bandwidth is commonly measured in terms of the ***half-power-point bandwidth*** of the gain factor given by

$$B_r = f_2 - f_1 \qquad \text{where } |H(f_1)|^2 = |H(f_2)|^2 = \tfrac{1}{2}|H(f_r)|^2 \tag{1.58}$$

For the usual case where the damping ratio is small, it can be shown by substituting Equation 1.58 into Equation 1.55a that

$$B_r \approx 2\zeta f_r \tag{1.59}$$

1.3.4 Multiple-Degree-of-Freedom Systems

Some physical systems can be easily modeled as a collection of discrete elements (lumped parameters) forming a multiple-degree-of-freedom (MDOF) system, meaning more than one coordinate is needed to describe the dynamic response of the system. This is particularly true of electrical systems composed of discrete circuit elements (e.g., inductors, capacitors, and resistors). It is also true of some mechanical systems that can be approximated by a collection of lumped masses, springs, and dampers. The analysis of lumped-parameter MDOF mechanical systems is thoroughly treated in most textbooks on mechanical vibration (e.g., [1.5]), and special cases with nonlinear elements are addressed in Chapter 13. For more general engineering applications, however, the ultimate MDOF mechanical system with distributed mass, stiffness, and damping is of greater interest, as discussed in the next section.